The oscilatory modes of a magnetically twisted compressible flux tube embedded in a compressible magnetic environment are investigated in cylindrical geometry. The general dispersion equation in terms of Kummer’s functions is obtained for the approximation of weak and uniform internal twist. The sausage, kink and fluting modes are examined by means of the derived exact dispersion equation. The solutions of this dispersion equation are found analytically for short and long wavelength limits under plasma conditions representative of the solar photosphere and corona. Numerical solutions for the phase velocity of the allowed eigenmodes are obtained for a wide range of wavenumbers and varying magnetic twist. Our results generalize previous classical and widely applied studies of MHD wave oscillations in magnetic loops with no twist. Applications to solar magneto-seismology are discussed.

Derivation of General Dispersion Equation

The plasma motion is governed by the system of single-fluid, linearized, ideal-MHD equations for a compressible magnetized plasma (see, e.g., [1]):

\[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \frac{\mathbf{B}}{\mu_0} \times (\nabla \times \mathbf{B}) + \frac{\mathbf{b}}{\rho_0} \times (\nabla \times \mathbf{B}) = 0, \]

\[ \nabla \cdot \mathbf{v} = 0, \]

\[ \rho_0 \frac{\partial V}{\partial t} = \nabla \cdot \mathbf{F}, \]

\[ \rho_0 \frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{B} \times \mathbf{v}), \]

\[ \nabla \cdot \mathbf{b} = 0, \]

where \( \mathbf{b} = \mu_0 \mathbf{B} \) is the magnetic field. The unperturbed state \( \mathbf{v}_0, \mathbf{b}_0, \rho_0, V_0, \mathbf{B}_0 \) is the equilibrium, where \( \rho_0 \), \( \mu_0 \), \( V_0 \), \( \mathbf{b}_0 \), \( \mathbf{B}_0 \) are the unperturbed density, magnetic permeability, and magnetic field. The unperturbed state and the geometry of the imple-

\[ \dot{B}_0 = \nabla \times (\mathbf{B}_0 \times \mathbf{v}_0), \]

\[ \dot{\rho}_0 = \nabla \cdot \mathbf{v}_0, \]

\[ \dot{V}_0 \]

Solution outside the flux tube

Outside the tube, \( \mathbf{v} = 0 \), \( \mathbf{b} = 0 \), \( \rho = \rho_0 \), \( V = V_0 \), \( \mathbf{B} = \mathbf{B}_0 \). The sound and Alfvén speeds are the same as \( \mathbf{v} = 0 \), \( \mathbf{b} = 0 \), \( \rho = \rho_0 \), \( V = V_0 \), \( \mathbf{B} = \mathbf{B}_0 \) for the internal Lagrangian displacement \( \xi_k(t) \):

\[ \xi_k(t) = \mathbf{B}_0 \times \mathbf{v}_0 \times \mathbf{v}_0 \mathbf{b}_0 \times \mathbf{B}_0 = \frac{1}{2} \mathbf{v}_0 \times \mathbf{v}_0 \mathbf{b}_0 \times \mathbf{B}_0. \]

The dispersion equation and solutions

Applying boundary conditions to the inside and outside solutions, yields the required general dispersion relation:

\[ D_0 \frac{\partial \xi_k}{\partial t} = D_{i} \frac{\partial^2 \xi_k}{\partial t^2}, \]

\[ D_0 \frac{\partial \xi_k}{\partial t} = D_{i} \frac{\partial^2 \xi_k}{\partial t^2}. \]

The dash denotes the derivative of a Kummer function evaluated at \( z = z_{i0} \), \( z_{i0} \) being the normalized variable, \( \xi_k = k(1 - \alpha k)^{1/2} \) is the effective longitudinal wavenumber. Let

\[ p_{02} \sim \mathcal{A} f(x), \]

where \( \lambda \) is an arbitrary constant and \( f(x) \) is an unknown function. If we choose \( \lambda = -1/2 \), the equation for total pressure takes the form

\[ \frac{d^2f}{dx^2} + \frac{1}{x} \frac{df}{dx} \left[ \lambda + \frac{1}{2} \frac{d}{dx} \right] f = 0, \]

where \( \alpha = \frac{m^2 - 4k^2}{2} = \mu_0 \mathbf{B}_0 \mathbf{b}_0 = 0 \), \( \mu_0 \mathbf{B}_0 \mathbf{b}_0 \approx 0 \), \( \mu_0 \mathbf{B}_0 \mathbf{b}_0 \approx 0 \). The solutions are known as the linear combination of Whittaker functions:

\[ f(x) = C_1 W_{m+1}^1 (x) + C_2 W_{m-1}^1 (x), \]

\[ W_{m+1}^1 (x) = e^{-x/2} M_{m+1/2} (x) + e^{x/2} M_{m-1/2} (x), \]

where \( C_2 \) is the arbitrary constant. The solutions can also be written in the form of Kummer functions [3]:

\[ W_{m+1}^1 (x) = e^{-x/2} M_{m+1/2} (x) + e^{x/2} M_{m-1/2} (x), \]

\[ W_{m-1}^1 (x) = e^{-x/2} M_{m-1/2} (x) + e^{x/2} M_{m+1/2} (x). \]

Fig. 5: The normalized phase-speed \( V_{ph} \) of the m=1 modes in a twisted tube (\( V_{ph} = 0.1 \)) as function of \( \xi_0 \) for cases \( \xi_0 > V_{ph} \) and \( \xi_0 > V_{ph} \). The external twist \( \xi_0 \) is shown at the left and right panels. The surface and body modes are shown. Note the change of character, from body to surface wave, at \( \xi_0 = 0 \) (left panel) and \( \xi_0 = 1 \) (right panel).

References