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Nonlinear Dynamics of Mirror Instability Revisited

O. A. Pokhotelov*, R. Z. Sagdeev†, M. A. Balikhin*, V. N. Fedun** and G. I. Dudnikova‡

*Automatic Control Systems Engineering, University of Sheffield, Sheffield, United Kingdom
†Department of Physics, University of Maryland, College Park, Maryland, USA
‡Department of Applied Mathematics, University of Sheffield, Sheffield, United Kingdom

Abstract. A nonlinear dynamics of the mirror modes near the instability threshold is revisited. It is shown that the major saturation is provided by modification of the velocity distribution function in the vicinity of small parallel ion velocities. The final relaxation scenario is based on almost resonant particle interaction with mirror modes. The saturated plasma state can be considered as a magnetic counterpart to electrostatic Bernstein-Greene-Kruskal modes. Our analytical model is verified by relevant numerical simulations. Test particle and PIC simulations indeed show that it is a modification of distribution function at small parallel velocities that results in fading away of free energy driving mirror mode. The physical similarity of the mirror and Weibel instabilities is demonstrated. The multipoint satellite measurements in space plasma can be used to validate a proposed scenario.

Keywords: Space plasmas, mirror waves, instability, kinetic effects
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INTRODUCTION

Mirror modes have been studied since the late 50’s when they were shown to be the result of the so-called mirror instability (MI), displaying the interplay between the magnetic pressure, the bulk plasma pressure and the pressure of resonant ions with almost zero parallel velocity [1]. Subsequent observations have shown them to be ubiquitous in nature, often as solitary holes or peaks in the magnetic field [2]. Over the past half a century these observations have also raised a number of intriguing conundrums such as the occurrence of mirror dips in regions of mirror stable plasma, the lack of mirror modes between peak structures even though the plasma is mirror unstable or the violation of adiabaticity in the ion temperatures. Recently the data from the THEMIS satellites were used to resolve these dilemmas in terms of the global structure of mirror modes and the role of the trapped particles in their dynamics [3, 4].

One of the first attempts of a nonlinear treatment of the mirror instability (MI) was made more than four decades ago [5]. The authors of this paper using the random phase approximation have reduced the problem of nonlinear saturation of the MI to the study of a quasilinear (QL) diffusion equation for the ion distribution function. Indeed, the background ion distribution function is shown to be modified which leads to saturation of MI. The important conclusion has been made on the special role of ions having small parallel velocities.

Some effects related to nonlinear saturation of mirror waves in the magnetosheath...
have been already discussed [6, 7]. A new nonlinear theory of MI in bi-Maxwellian plasmas has been proposed where the formation of magnetic holes has been suggested in terms of a process known under the name of wave collapse [8]. A further development of such scenario was offered in [9]. The role of trapped particles in the MI nonlinear dynamics was discussed in [10, 11, 12]. Recently the model describing the matching the QL theory for the space-averaged ion distribution function with a reductive perturbative description of the mirror modes has been presented [13]. The model was based on the numerical modelling of the diffusive equation for the ion distribution function. The purpose of the present manuscript is to provide a further nonlinear analysis of the MI which, however, is based on the exact analytical solution of the diffusion equation. We will show that flattening of the ion distribution function which is inherent to the QL dynamics substantially modifies the mirror mode dispersion equation and results in two important effects. The first one is connected with the fact that the resonant interaction of the ions with the mirror modes rapidly vanishes and gives the place for the weaker adiabatic interaction, similar to the Bernstein-Greene-Kruskal (BGK) modes [14]. As the result the MI dispersion relation becomes of the second order equation in frequency. The second effect is associated with the additional decrease in the free energy which turns out to be much stronger than that predicted by reductive perturbation model.

GENERAL DISPERSION RELATION

We consider a collisionless plasma composed of ions and electrons embedded in a magnetic field \( \mathbf{B} \). We make use of a Cartesian coordinate system in which the unperturbed magnetic field \( \mathbf{B}_0 \) is directed along the \( z \) axis, the \( x \) axis is along the wave vector and the \( y \) axis to complete the triad. The total magnetic field is \( \mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B} \), where \( \delta \mathbf{B} \) corresponds to perturbation. The perturbation of the magnetic field consists of only \( z \) and \( x \) components satisfying the property of solenoidality, i.e. \( \nabla \cdot \delta \mathbf{B} = \partial \delta B_z / \partial x + \partial \delta B_x / \partial z = 0 \). The \( \delta B_z \) component corresponds to the so-called non-coplanar magnetic component, and does not enter our basic equations and thus can be set to zero. Furthermore, our analysis will be limited to the case of most importance when the ion temperature is much greater than the electron temperature. Assuming all perturbed values to vary as \( \sim \exp(-i \omega t + i k \cdot \mathbf{r}) \) for each \((k, \omega)\) mode the linear response of the distribution function to the mirror-type perturbations is

\[
\delta F_k = \frac{v^2}{2} \left( \frac{\partial F}{v^\parallel \partial v^\parallel} - \frac{\partial F}{v^\perp \partial v^\perp} \right) b_k - \frac{v^2}{2} \omega - k^\parallel v^\parallel \frac{\partial F}{\partial v^\parallel} b_k, \tag{1}
\]

where \( F \) is unperturbed ion distribution function and \( \delta F_k \) corresponds to the Fourier component of the perturbation, \( v^\perp(||) \) the perpendicular (parallel) particle velocity, \( k^\parallel \) the component of the wave vector \( k \) parallel to the external magnetic field \( \mathbf{B}_0 \), \( \omega \) the wave frequency and \( b_k = \delta B_z(k)/B_0 \) the dimensionless amplitude of the \( k^{th} \) harmonic of the perturbation.

The first term on the right-hand side of (1) is the so-called mirror force (due to that the instability is called mirror instability (MI)), it vanishes if plasma is isotropic. The second term refers to the kinetic contribution. The MI is found in the low-frequency

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limit when $\omega \ll k v_\parallel$. In this limit the second term in (1) is negligible except for particles with $v_\parallel = 0$. For these particles this term is of the same order and potentially of larger magnitude than the preceding mirror term. The expansion of the resonant denominator in this case reads

$$\frac{\omega}{\omega - k v_\parallel} \approx -\frac{i\pi \omega}{|k||}\delta(v_\parallel) + \frac{\omega^2}{k^2 v_\parallel},$$

(2)

where $\delta(x)$ is the Dirac delta function. Usually the second term in (2) is neglected as the small parameter of the order of $\omega/k v_\parallel \approx K$, where $K$ is the instability threshold. However, in the nonlinear regime the character of the expansion is changed, instead of small parameter $\omega/k v_\parallel$ we will have $\omega/k v_\parallel \Delta v^*$, where $\Delta v^*$ is the width of the zone occupied by the flattening of the ion velocity distribution function and thus the second term in Eq. (2) can be important or even dominate.

With the help of expansion (2) the perturbation of the distribution function reads

$$\delta F_k = \frac{v_\perp^2}{2} \left( \frac{\partial F}{v_\parallel \partial v_\parallel} - \frac{\partial F}{v_\perp \partial v_\perp} \right) b_k - \frac{v_\perp^2}{2} \frac{\omega^2}{k^2} \frac{\partial F}{v_\parallel^3 \partial v_\parallel} b_k$$

$$+ i\pi \frac{\omega}{k} \delta(v_\parallel) \frac{\partial F}{v_\parallel \partial v_\parallel} b_k.$$

(3)

The magnetic mirror mode is the pressure balanced structure and thus obeys the perpendicular plasma pressure condition [10]

$$\frac{\delta p_\perp}{2p_{\perp,0}} + \frac{1}{\beta_\perp} \left( 1 + \frac{3}{4} \rho^2 k_\perp^2 \right) b_k = -\frac{k^2}{k^2 \beta_\perp} \left( 1 + \beta_\perp - \beta_\parallel \right) b_k$$

(4)

where $p_{\perp,0}$ is the equilibrium perpendicular plasma pressure and $\delta p_\perp = \int (mv_\perp^2/2) \delta F_k dv$ corresponds to its variation. Using (3) and calculating $\delta p_\perp$ from Eq. (4) one obtains

$$\left[ \Delta + \frac{i\pi m}{8p_{\perp,0}} \frac{\omega}{|k|} \int v_\perp^4 \frac{\partial F}{v_\parallel \partial v_\parallel} \delta(v_\parallel) dv - \frac{\omega^2}{k^2} I_1 \right] b_k = 0$$

(5)

where

$$\Delta = \frac{m}{8p_{\perp,0}} \int v_\perp^4 \left( \frac{\partial F}{v_\perp \partial v_\perp} - \frac{\partial F}{v_\parallel \partial v_\parallel} \right) dv - \frac{1}{\beta_\perp} \left( 1 + \frac{3}{2} \rho^2 k_\perp^2 \right) - \frac{k^2}{k^2 \beta_\perp} \chi,$$

(6)

$$I_1 = \frac{m}{8p_{\perp,0}} \int v_\perp^4 \frac{\partial F}{v_\parallel^3 \partial v_\parallel} dv,$$

(7)

and $\chi = 1 + (\beta_\perp - \beta_\parallel)/2$.

Eq. (5) shows that in general the MA dispersion relation is the second order equation in frequency. Of course in the linear limit the additional term, containing $\omega^2$, is small.
as $\omega^2 / k_{\parallel}^2 v_{T\parallel}^2 \ll 1$ (where $v_{T\parallel}$ is the plasma characteristic thermal velocity) relative to the resonant term. However, as it will be shown in what follows in the nonlinear regime the expansion parameter $\omega / k_{\parallel} v_{T\parallel}$ is now replaced by $\omega / k_{\parallel} |\Delta u|$, where $\Delta u$ is the width of the "plateau" in the ion velocity distribution, which is much smaller than the ion thermal velocity. Due to that the quadratic term can be important or even dominate.

By integrating by parts the first term on the r.h.s. of (6) can be re-arranged as

$$\Delta = K - \frac{3}{4} \frac{\rho_{\perp}^2 k_{\perp}^2}{\beta_{\perp}} \left( 1 + \frac{\beta_{\perp} - \beta_{\parallel}}{2} \right), \quad (8)$$

where

$$K = I_2 - \frac{1}{\beta_{\perp}}, \quad (9)$$

and

$$I_2 = -\frac{m}{8 \rho_{\perp} v_{T\perp}} \int v_{\perp}^4 \frac{\partial F}{\partial v_{\parallel}} dv_{\parallel}. \quad (10)$$

The quantity $K$ represents the instability condition for the arbitrary velocity distribution function. In bi-Maxwellian plasma it reduces to the usual value, $K \equiv T_{\perp}/T_{\parallel} - 1 - \beta_{\perp}^{-1}$, where $T_{\perp(\parallel)}$ is the perpendicular (parallel) plasma temperature.

### FLATTENING OF THE ION DISTRIBUTION FUNCTION

For the sake of clarity we consider that due to the rapid motion of resonant particles we assume that in the vicinity of small parallel velocities the background ion distribution function would flatten and takes the shape of quasi-plateau. This to happen does not necessarily require the assumption of random phases and is valid even in the single-mode (sinusoidal) regime. In order to take into account the effect of flattening of the distribution function we assume that the coefficients in Eq. (5) are not frozen to their initial values and are evaluated from the instantaneous distribution function given by the QL diffusion equation. In QL regime the amplitude of oscillations remains so small that perturbations of particle velocities and particle densities are linear relative to the wave amplitude. Only the averaged distribution function slowly varies under chaotic wave perturbations. The equation that governs this slow variation has been derived in [5] and is

$$\frac{\partial F}{\partial t} = \frac{1}{2} \sum_k \gamma_k |b_k|^2 \left\{ \frac{1}{v_{\perp}} \frac{\partial}{\partial v_{\perp}} \left[ v_{\perp}^2 \left( \frac{v_{\perp}^2}{2} \frac{\partial F}{\partial v_{\perp}} - v_{\parallel}^3 \frac{\partial F}{\partial v_{\parallel}} \right) \right] \right.$$  

$$+ \left. \frac{\partial}{\partial v_{\parallel}} \left( v_{\parallel}^4 \frac{\partial F}{\partial v_{\parallel}} \right) \right\}. \quad (11)$$

The QL approximation is valid if $|k_{\|} | \Delta v_{\parallel} \gg \gamma_k$, where $\Delta v_{\parallel}$ is the the region occupied by diffusion. In our case $\Delta v_{\parallel} \approx v_{T\perp}$. Since $\gamma_k / |k_{\|} v_{T\parallel} \approx K$, where $K \equiv T_{\perp}/T_{\parallel} - 1 - \beta_{\perp}^{-1}$
is the instability threshold condition, the validity of QL approximation is satisfied when deviation of plasma parameters from the equilibria is not substantial.

Eq. (11) shows that the last term on the right-hand side possesses a strong singularity in the vicinity of $v_\parallel \to 0$ and thus the most noticeable change in the shape of the distribution function arises in this region. Therefore, Eq. (11) reduces to

$$\frac{\partial F}{\partial t} = \sum_k \gamma_k |b_k|^2 \frac{v_\perp^4}{4} \frac{\partial}{\partial v_\parallel} \left( \frac{\partial F}{v_\perp^2 \partial v_\parallel} \right).$$

(12)

With the help of the relation

$$\frac{\partial |b_k|^2}{\partial t} = 2 \gamma_k |b_k|^2,$$

(13)

Eq. (12) reduces to

$$\frac{\partial F}{\partial h} = \frac{v_\perp^4}{4} \frac{\partial}{\partial v_\parallel} \left( \frac{\partial F}{v_\perp^2 \partial v_\parallel} \right),$$

(14)

where

$$h = \sum_k |b_k|^2.$$

(15)

Eq. (14) can be easily solved by decomposing the variables and searching the solution in the form of Fourier-Bessel integral. We assume that for $h = 0$ the distribution function $F(h, v_\parallel, v_\perp)$ reduces to bi-Maxwellian form, i.e.

$$F_0(v_\perp, v_\parallel) = \frac{n}{\pi^{3/4} v_T^2 v_{T_\parallel}^2} \exp \left( -v_\perp^2/v_T^2 - v_\parallel^2/v_{T_\parallel}^2 \right),$$

(16)

where $n$ is the plasma density and $v_{T_\perp(\parallel)}$ is perpendicular (parallel) thermal velocity.

The result can be written as

$$F(h, v_\parallel, v_\perp) = C(v_\perp) |v_\parallel|^{3/2} \int_0^\infty \frac{-t^{3/2} \left( v_\parallel^2 / v_{T_\parallel}^2 \right)^{1/2} t^{3/2} J_{3/4} \left( tv_\parallel^2 / v_{T_\parallel}^2 \right)}{(t^2 + 1)^{3/4}} dt,$$

(17)

where $C(v_\perp) = \Gamma(3/4) n \exp(-v_\perp^2/v_T^2) / \pi^2 v_T^2 v_{T_\parallel}^{5/2}$, $\Gamma(x)$ and $J_\nu(x)$ are the Gamma and Bessel functions, respectively.

The plot of the distribution function $F$ as a function of $v_\parallel$ for different values of $h$ and constant $v_\perp$ is depicted in Fig. 1. One sees that velocity diffusion leads to substantial flattening of the distribution function for small $v_\parallel$. Instead of initial dependence $F \propto \exp(-v_\perp^2/v_T^2)$ the distribution function now scales as $\propto \exp(-v_\parallel^4/v_{T_\parallel}^4)$ and thus the second term on the left (the term containing the $\delta$-function) vanishes and the next term in the expansion (2) starts to play an important role.
FIGURE 1. Velocity distribution function \( F \) as the function of the wave amplitude \( h \).

**QL GROWTH RATE**

The first nonzero term will be that proportional to \( \omega^2 \) and the mirror dispersion relation becomes of the second order in frequency. Actually the flattening of the velocity distribution function mainly affects two terms in the dispersion relation containing the integrals \( I_1 \) and \( I_2 \). Using the explicit expression for the distribution function (17) one can easily calculate the value of \( I_1 \)

\[
I_1 \simeq -\frac{\alpha v^2}{v^2_{T\perp}} \left( \frac{T_{T\perp}}{T_{T\parallel}} \right)^{3/2} h^{-1/4}, \tag{18}
\]

where \( \alpha = \frac{3}{16} \Gamma(\frac{1}{4}) \Gamma(\frac{1}{2}) \simeq 1.2 \).

For calculation of \( I_2 \) it is more convenient to find the deviation from its equilibrium value, i.e. to find \( \delta I_2 = I_2 - I_{20} \), where

\[
I_{20} = -\frac{m}{8 p_{\perp 0}} \int v^4_{\perp} \frac{\partial F_0}{\partial v_{\parallel}} dv_{\parallel} = \frac{T_{T\perp} v_{\perp}}{T_{T\parallel} v_{\parallel}}. \tag{19}
\]

Then one finds

\[
I_2 = \frac{T_{T\perp}}{T_{T\parallel}} - \mu \left( \frac{T_{T\perp}}{T_{T\parallel}} \right)^{3/2} h^{1/4}, \tag{20}
\]

where \( \mu = -\frac{3}{72} \Gamma(-3/4) \Gamma(\frac{1}{4}) \simeq 1.6 \).

Substituting (18) and (20) into (5) one obtains the nonlinear growth rate
\[ \gamma = \frac{|k||v||}{\alpha^2} \left( \frac{T_\parallel}{T_{\perp}} \right)^{1/4} h^{1/8} \left( K - \frac{3}{4\beta_{\perp}} \rho_i^2 k_{\perp}^2 - \mu \left( \frac{T_\perp}{T_{\parallel}} \right)^{3/2} h^{1/4} - \frac{k_{\parallel}^2}{k_{\perp}^2} \beta_{\parallel} \right)^{1/2}. \] (21)

Let us estimate the growth rate of the most growing mode. We consider that in the course of QL evolution the wave amplitude is so small that we do not deviate from the initial most growing mode. The latter corresponds to the values of the parallel and perpendicular wave numbers given by the linear theory [10]

\[ (k_{||}\rho_i)^2_{\text{max}} = \beta_{||}^2 \frac{K^2}{12\chi}. \] (22)

and

\[ (k_{\perp}\rho_i)^2_{\text{max}} = \beta_{\perp} \frac{K}{3}. \] (23)

Thus, the nonlinear growth rate for this mode is

\[ \gamma_{\text{max}} = \frac{\omega_{ci}}{2^{3}\frac{\alpha^2}{\chi^2}} \left( \frac{T_\parallel}{T_{\perp}} \right)^{3/4} h^{1/8} K \left( K - 2\mu \left( \frac{T_\perp}{T_{\parallel}} \right)^{3/2} h^{1/4} \right)^{1/2}. \] (24)

From (24) follows that QL saturation of the most growing mode is attained at relatively small amplitude when \( K = 2\mu \left( T_{\perp}/T_{\parallel} \right)^{3/2} h^{1/4} \) or

\[ \delta B_z/B_0 \approx K^2 \left( T_{\parallel}/T_{\perp} \right)^3/4. \] (25)

At this point a few comments are in order. The further nonlinear evolution of the MI will be under control of the effects that were excluded from our analysis. Among them are the mode coupling (incurred stresses) and nonlinear variations of the FLR effect. The departure is observed when the QL evolution tends to saturate the magnetic field fluctuations. If the mode coupling is taken into consideration the magnetic energy continues to grow and \( b \) displays a sharp increase leading to a finite-time blow-up, in accordance with the model of subcritical bifurcation [8]. The termination of the singularity requires incorporation of the nonlinear variations of the FLR effect.

**SUMMARY**

We have presented a local analysis of the MI in a high-\( \beta \) non-Maxwellian plasma taking into account the effect of flattening of the ion velocity distribution function near the instability threshold. QL evolution of the MI was investigated by direct integration of the corresponding diffusion equation. It has been shown that due to the fattening of the ion distribution function the resonant interaction of the ions with \( v_{||} \approx 0 \) is rapidly "switched off" and then replaced by a weaker adiabatic interaction with mirror mode.
At this stage the mirror mode behaves similar to the BGK mode [14]. This fact was not appreciated in the previous analyses. The MI dispersion relation which in the linear regime is a differential equation of the first order in time derivative now becomes of the second order. Furthermore, it has been shown that the main decrease in the free energy, necessary for the instability, is due to the modification of the ion velocity distribution which is very subtle near the instability threshold. It should be mentioned that during linear and QL stage the MI evolution mathematically similar to another instability, the Weibel instability [15] which can be described by nearly the same differential equations. The details will be published in a separate publication. The differences arise when one incorporates the higher order nonlinearities. For the MI they are quadratic in the wave amplitude whereas for the Weibel instability they are cubic. This results in different saturated states. In the first case they appear as solitary waves whereas for the Weibel instability they form the filamentary structures [16].

The model developed in our paper still remains oversimplified. For example, it has been restricted to the case of isotropic electrons when \( |b| < 1 \). The case when \( \delta B \sim B \) was described in [17]. Furthermore, the effect of bistability of mirror modes revealed in recent observations and discussed in some papers [9, 17] was also outside the scope of this study. However, our analysis has provided a deeper insight into the physics of nonlinear dynamics of mirror modes in high-\( \beta \) space plasmas.

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