Generation of short-lived large-amplitude magnetohydrodynamic pulses by dispersive focusing

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Abstract

Large-amplitude MHD waves are routinely observed in space plasmas. We suggest that dispersive focusing, previously proposed for the excitation of freak waves in the ocean, can be also responsible for the excitation of short-lived large-amplitude MHD waves in space plasmas. The DNLS equation describes MHD waves propagating in plasmas at moderate angles with respect to the equilibrium magnetic field. We obtained an analytical solution of the linearised DNLS equation governing the generation of large-amplitude MHD waves from small-amplitude wave trains due to the dispersive focusing. Our numerical solutions of the full DNLS equation confirm this result.

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1. Introduction

Freak (or rogue) waves have been the subject of intensive investigations in oceanography during the last decade [1–4]. In the first studies of this phenomenon the Korteweg–de Vries (KdV) and Nonlinear Schrödinger (NLS) equations were used to describe the nonlinear wave dynamics. Later the theory was extended to two-dimensional model equations, and also numerical modelling using full hydrodynamic description has been carried out.

Recently, the generation of large-amplitude waves described by the modified Korteweg–de Vries (mKdV) equation has been investigated [5]. The mKdV equation describes internal waves in stratified fluids [6] as well as magnetohydrodynamic (MHD) waves in plasmas [7]. Thus, large-amplitude MHD waves can be generated from small-amplitude wave trains in plasmas.

The applicability of the mKdV equation to MHD waves is restricted to a special class of MHD waves where the component of magnetic field normal to the direction of wave propagation remains almost constant. MHD waves observed in space missions very rarely satisfy this condition. Therefore it is desirable to extend the analysis carried out in [5] to a more general class of MHD waves. MHD waves propagating at relatively small angles with respect to the equilibrium magnetic field are routinely observed in the solar wind by space missions [8]. The normal component of the magnetic field varies in these waves in a wide range, from zero to about 30% of the equilibrium magnetic field. The waves of this type are described by the Derivative Nonlinear Schrödinger (DNLS) equation.

In this Letter we study the generation of short-lived large-amplitude MHD pulses by dispersive focusing described by the DNLS equation. The Letter organised as follows. In the next section we consider analytically the generation of large-amplitude MHD pulses using the linearised DNLS equation. In Section 3 we present the results of numerical modelling of large-amplitude MHD pulse generation described by the full nonlinear DNLS equation. Section 4 contains the summary of the results and our conclusions.

2. Linear theory of large-amplitude MHD pulse generation

To describe the MHD wave propagation we use the DNLS equation [9–13]:

$$\frac{\partial b}{\partial t} + \alpha \frac{\partial}{\partial x} (|b|^2 b) + ik \frac{\partial^2 b}{\partial x^2} = 0. \quad (1)$$

Here \(b = B_y + iB_z\), where \(B_y\) and \(B_z\) are the \(y\) and \(z\)-components of the magnetic field,

$$\alpha = \frac{V_A}{4B_x^2(1 - \beta)}, \quad \kappa = \frac{V_A^2}{2\Omega_i}, \quad V_A = \frac{B_x^2}{\mu_0 \rho_0}, \quad \Omega_i = \frac{eB_x}{m_i},$$

\(B_x\) is the \(x\)-component of the magnetic field (which remains constant), \(\beta\) is the square of the ratio of the sound speed to the Alfvén speed \(V_A\), and \(\Omega_i\) is the ion gyrofrequency; \(\mu_0\), \(e\) and \(m_i\) are the magnetic permeability of free space, the elementary charge, and the ion mass, respectively. Note that Eq. (1) is written in the reference frame moving with the speed \(V_A\) in the positive \(x\)-direction with respect to the rest plasma. Note that the DNLS equation has...
been derived under the assumption that the nonlinearity and dispersion are small. Hence, it only describes waves with the amplitude of the magnetic field perturbation that is sufficiently smaller than the equilibrium magnetic field, and with the wavelength sufficiently larger than the ion inertia length.

In this Letter we explore the generation of large-amplitude MHD wave pulses from small-amplitude wave trains using the DNLS equation (1). We restrict our analysis to waves propagating exactly along the equilibrium magnetic field, so that \( b \) satisfies the boundary condition \( b = 0 \) as \(|x| \to \infty\).

We start the analysis from considering the linear approximation. The linearised DNLS equation is

\[
\frac{\partial b}{\partial t} + ik \frac{\partial^2 b}{\partial x^2} = 0.
\]

This is a diffusion equation, but with a purely imaginary diffusion coefficient \(-ik\). The solution of the initial value problem vanishing as \(|x| \to \infty\) can be obtained from the corresponding solution for the diffusion equation (see, e.g., Ref. [14]) by substituting \(-ik\) for the diffusion coefficient:

\[
b(t,x) = b(0,x) e^{i(kx−\omega t)}.
\]

Let us first consider the evolution of an amplitude-modulated wave packet,

\[
b_0(x) = \alpha \exp\left(-\frac{(x/l)^2}{2}\right),
\]

where \(\alpha > 0\), \(l\) and \(\lambda\) are real constants. Since we are interested in the generation of large-amplitude pulses, we only need to focus on \(|b|\) in what follows. Substituting (4) in (3) we obtain

\[
|b| = \frac{\alpha l}{[l^2+16\kappa^2 t^2]^1/4} \exp\left(-\frac{2\alpha^2 + 2\kappa^2 t^2}{l^2+16\kappa^2 t^2}\right).
\]

At fixed \(t > 0\), \(|b|\) takes its maximum value at \(x = -2\kappa l t/l\), and this value monotonically decreases with time. Hence, the generation of large-amplitude MHD waves from a small-amplitude wave train given by (6) is impossible.

To give a physical insight in the process of large-amplitude wave excitation, let us consider the initial condition in the form of an amplitude and frequency modulated wave train,

\[
b_0(x) = \alpha \exp\left(-1 + i\xi (x/l)^2\right),
\]

where \(\xi\) is a real constant. Substituting (6) in (3) we arrive at

\[
|b| = \frac{\alpha l}{[l^2+4\kappa^2 (\xi)^2+(4\kappa^2 t^2)]^{3/4}} \exp\left(-\frac{\lambda^2}{l^2+4\kappa^2 (\xi)^2+(4\kappa^2 t^2)}\right).
\]

Obviously, at fixed \(t > 0\), \(|b|\) takes its maximum value at \(x = 0\), and this maximum is given by

\[
|b|_{\text{max}} = \frac{\alpha l}{[l^2+4\kappa^2 (\xi)^2+(4\kappa^2 t^2)]^{1/4}}.
\]

When \(\xi < 0\), \(|b|_{\text{max}}\) is a monotonically decreasing function of time, so large-amplitude pulses cannot be generated. However, when \(\xi > 0\), the situation is different. Now \(|b|_{\text{max}}\) takes its maximum value at

\[
t = t_{\text{max}} = \frac{\xi^2}{4\kappa(1+\xi^2)}.
\]

and this maximum is equal to \(\alpha(1 + \xi^2)^{1/4}\). When \(\xi \gg 1\), the ratio of the wave amplitude at \(t = t_{\text{max}}\) and \(t = 0\) is approximately equal to \(\sqrt{\xi} \gg 1\), i.e. large-amplitude pulses can be generated from small-amplitude wave trains.

Let us now take \(x = x_0 + \delta x\), \(|x_0| \gg 1/|\xi|\), and consider \(|\delta x| \ll |x_0|\) assuming that \(|\xi| \gg 1\). Then (6) can be rewritten in the approximate form as

\[
b_0 = A e^{i \xi x_0} + \kappa_0 = 2\xi x_0/l^2, \quad A = a \exp\left(-1 + i\xi \right)(x_0/l)^2.
\]

These expressions represent a circularly polarised monochromatic wave with constant amplitude \(A\). This wave is left-hand polarised when \(k_0 > 0\) (\(\xi x_0 > 0\)), and right-hand polarised when \(k_0 < 0\) (\(\xi x_0 < 0\)). Hence, a large-amplitude wave can be generated from a small-amplitude wave train given by (6) when this wave train is locally left-hand polarised for \(x > 0\) and right-hand polarised for \(x < 0\), and cannot be generated in the opposite case. Note that a circularly polarised wave with constant amplitude is an exact solution of the DNLS equation. When the amplitude of such a wave is sufficiently small, this wave is unstable with respect to paral-\(\text{lel modulations when it is left-hand polarised}[10,12]\). In contrast, a right-hand polarised wave is always stable with respect to parallel modulations.

When \(\xi \gg 1\), the pulse amplitude is larger than a half of its maximum value, i.e. \(|b|_{\text{max}} > \frac{\alpha}{\sqrt{2}}\) only in a narrow time interval determined by

\[
[t - t_{\text{max}}] < \frac{l^2}{2k^2\xi} = T.
\]

For \([t - t_{\text{max}}] \gtrsim t_{\text{max}}\), it follows from (8) that the characteristic time of variation of \(|b|_{\text{max}}\) is \(l^2/k \gg T\). This estimate reveals the short-lived character of larger-amplitude pulses, similar to ones found for perturbations described by the KdV and mKdV equations [2,5].

There is a very simple qualitative explanation why large-amplitude pulses can be generated from initial perturbations given by (6), and cannot be generated from initial perturbations given by (4). If \(\lambda \lesssim 1\) in (4), then the diffusion of the wave energy dominates dispersion. When \(\lambda \gg 1\), the perturbation given by (4) is a quasi-monochromatic wave, so the dispersive focusing is im-\(\text{possible. On the other hand, the initial perturbation given by (6) contains a wide spectrum of harmonics, so that the mechanism of dispersive focusing works and, for } \xi \gg 1\), it dominates the wave energy diffusion.

The qualitative explanation of the fact that the generation of large-amplitude waves from the initial small-amplitude wave train given by (6) is only possible when \(\xi > 0\) is equally simple. This explanation is based on the local representation given by (10). It follows from (2) and (10) that the local dispersion equation takes the form \(\omega = -k^2_0\), so that the local group velocity is \(v_x = -2k_0\). Then, for \(\xi > 0\), \(v_x > 0\) when \(x > 0\), and \(v_x < 0\) when \(x < 0\) (recall that we use the reference frame moving with the speed \(v_\lambda\) in the positive \(x\)-direction with respect to the rest plasma). This implies that, at the initial moment of time, the energy flux is directed towards the coordinate origin, which makes the wave amplitude at the coordinate origin growing. On the other hand, for \(\xi < 0\), \(v_x > 0\) when \(x > 0\), and \(v_x < 0\) when \(x < 0\). The energy flux is directed outwards from the coordinate origin, and as a result, the wave amplitude at the coordinate origin decreases.

Using this qualitative picture we can estimate the time when the wave energy contained in a small portion of the wave train near \(x_0\) arrives at the coordinate origin. This time is equal to \(|x_0/v_x| = l^2/4k^2\xi\). Amazingly, it is exactly equal to \(t_{\text{max}}\) given by (9) for \(\xi \gg 1\).

3. The effect of nonlinearity

To study the effect of nonlinearity on the wave dispersive focusing we solved the DNLS equation (1) with the initial condition (6) numerically. We carried out computations for \(\xi > 0\) only. The DNLS equation is a completely integrable equation and can be solved by the inverse scattering method (ISM) [15–18]. The ISM is very
appropriate for calculating the asymptotic behaviour of solutions with arbitrary initial conditions. However, its applicability to studying the intermediate behaviour of solutions is restricted to the initial conditions for which the corresponding scattering problem can be solved analytically. To our knowledge, the only non-trivial exact solution to the DNLS equation obtained so far is the $N$-soliton solution [19,20]. It is rather less probable that the scattering problem for the DNLS equation for the initial condition (6) can be solved analytically. This observation inspired us to solve the DNLS equation (1) with the initial condition (6) numerically. The evolution of the initial perturbation strongly depends on the value of the nonlinearity parameter $N$ given by

\[ N = \frac{\alpha a^2}{\kappa}. \]

This parameter characterises the relative importance of nonlinearity and dispersion. Introducing dimensionless variables

\[ q = \frac{b}{a}, \quad X = \frac{x}{\ell}, \quad T = \frac{\kappa t}{\ell^2}, \]

we rewrite Eq. (1) in the dimensionless form

\[ \frac{\partial q}{\partial T} + \frac{\partial}{\partial X} \left( |q|^2 q \right) + \frac{i}{\ell^2} \frac{\partial^2 q}{\partial X^2} = 0. \]

(11)

The initial condition (6) recasts

\[ q = \exp \left( -1 + i \xi \right) X^2 \]

at $T = 0$. (12)

Eq. (11) with the initial condition (12) was solved using the pseudo-spectral method. Fig. 1 shows the dependence of the ratio of maximum amplitude to the initial amplitude, $|b|_{\max}/a$, on $\xi$ for different values of $N$. We see that nonlinearity decreases the maximum amplitude and thus it works against the dispersive focusing.

In Figs. 2 and 3 the evolution with time of the $X$-dependence of $|q|$ is shown for $N = 0$ and $N = 0.45$, respectively. These figures reveal the short-living nature of large-amplitude pulses. It is clearly seen in Fig. 3 that nonlinearity not only reduces the efficiency of large-amplitude pulse generation from the initial perturbation given by (12), but also causes the pulse to split. For sufficiently large time ($T \gtrsim 20$) there are two maxima in the wave profile, one large and one small. This is a standard behaviour in soliton equations when the initial perturbation is not a soliton.

The numerical results presented in Fig. 1 make the impression that the nonlinearity plays a negative role in the large-amplitude pulse generation. However in fact, these results only show that the nonlinearity suppresses the large-amplitude pulse generation form the particular initial perturbation given by (12). It is quite possible that large-amplitude pulses can be generated from small-amplitude initial perturbations even in the nonlinear regime if we change the initial condition. To verify this conjecture we did the following. Let $q_{\text{lin}}(T, X)$ be the solution of (11) with $N = 0$ and with the initial condition (12). This solution takes its maximum amplitude $(1 + \xi^2)^{1/4}$ at $T = T_{\max} = \kappa t_{\max}/\ell^2$. We solved (11) with $N = 0.45$ taking $q_{\text{lin}}(T_{\max}, X)$ as the initial condition, and integrating it backward with respect to time. As a result, we obtained the solution of (11), $q_{\text{nonl}}(T, X)$, for $T < t_{\max}$. We found the moment of time, $T = T_{\min} < T_{\max}$, when the amplitude of $q_{\text{nonl}}(T, X)$ takes its minimum value. The function $q_{0}(X) = q_{\text{nonl}}(T_{\min}, X)$ is shown in Fig. 4. Then we solved (11) with $N = 0.45$ and with the initial condition $q = q_0$ at $T = 0$. As a result we obtained $q = q_{\text{lin}}(T_{\max}, X)$ at $T = T_{\max} - T_{\min}$. We carried out this calculations for $\xi = 6$.

It is clearly seen in Fig. 4 that the amplitude of $q_{0}(X)$ is smaller than unity. Hence, the amplification rate obtained in the nonlinear regime with the initial perturbation $q_0$ is even larger than the amplification rate obtained in the linear regime with the initial perturbation (12). This example clearly shows that the dispersive focusing works both in nonlinear as well as in linear regime.

4. Summary and conclusions

In this Letter we have studied the generation of large-amplitude MHD pulses by dispersive focusing. We have used the derivative nonlinear Schrödinger equation (DNLS) to describe the wave propagation. We started from the linearised equation and solved the initial value problem analytically. We have shown that large-amplitude pulses cannot be generated from amplitude-modulated...
Fig. 4. The initial perturbations used in linear \((N=0)\) and nonlinear \((N=0.45)\) calculations. The thick and thin solid curves show \(|q|\) and \(\text{Re}(q)\) respectively. The left curves correspond to \((12)\) with \(\xi=6\). Since equation \((11)\) is invariant with respect to shift of \(X\) the spatial positions of the initial perturbations are chosen arbitrarily.

quasi-harmonic wave trains. On the other hand, large-amplitude pulses can be generated from amplitude and frequency-modulated wave trains. An intrinsic property of this pulses is that they are short-lived. Exactly, this means that the characteristic time of the large-amplitude pulse existence is much smaller than the characteristic time of the initial perturbation evolution, at least when we consider moments of time not close to the time of pulse appearance.

To study the effect of nonlinearity we solved the DNLS equation numerically. First we took the same initial condition as in the linear case. It turned out that, for this initial condition, the nonlinearity reduces the efficiency of large-amplitude pulse generation. However, the large-amplitude pulses can be generated even more efficiently in the nonlinear case than in the linear one if the initial condition is slightly modified.

Hence, summarising the results obtained in the Letter, we conclude that the dispersive focusing is a very efficient mechanism of generation of large-amplitude pulses of MHD waves in space plasmas. We should specially emphasise that these pulses are short-lived. They are generated from small-amplitude perturbations and quickly disappear. Since special initial perturbations are needed for their generation, these pulses should be relatively rare events.

In conclusion, we once again reiterate that the DNLS equation used to describe the MHD waves in this Letter is only valid for small-amplitude long waves propagating at small angles with respect to the equilibrium magnetic field. For the description of waves with arbitrary amplitudes propagating in arbitrary direction the full system of the Hall MHD equations has to be used.

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References